Int. J. Solids Structures, 1968, Vol. 4, pp. 757 to 768. Pergamon Press. Printed in Great Britain

THE NON-LINEAR PERTURBATION ANALYSIS OF DISCRETE STRUCTURAL SYSTEMS

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Abstract—A general form of perturbation analysis for discrete non-linear structural systems is presented. This generates a system of linear equations which can be solved sequentially for the path derivatives in the unloaded state. Each set of linear equations has the same basic non-singular matrix, and the method is thus ideally suitable for use with a digital computer.

The general theory is illustrated firstly by an harmonic and secondly by a finite element analysis of a beam suffering large bending deflections, an exact beam formulation being employed and a continuum perturbation analysis being presented for comparison. The first seven path derivatives are evaluated and are observed to converge rapidly in each case to the continuum values which are then used to construct the load–deflection characteristic of the beam: the choice of independent variable in this *final* construction is seen to be highly significant.

Good agreement is achieved with the known non-linear solution, and it is concluded that the perturbation analysis will be a useful tool in problems of moderate non-linearity.

1. INTRODUCTION

THE discrete representation of continuous structural systems, using for example the finite-element Rayleigh–Ritz procedure, is an established tool in the analysis of practical solids and structures. Much work has been done in the small-deflection linear field, and recent interest has focused on large-deflection non-linear problems with their associated stability questions.

Aside from this analytical activity, discrete structural systems have been studied in their own right, and a general non-linear theory has been developed by Thompson [1, 2] and Sewell [3, 4], this theory being primarily aimed at the study of path and stability phenomena. In the development of these general studies the intrinsic determination of equilibrium path derivatives has been systematically employed, and it is the purpose of the present paper to illustrate the *analytical* advantages of this procedure. The resulting method of analysis corresponds to the perturbation techniques of continuum mechanics, although the intrinsic formulation contains no suggestion of a perturbation. The non-linear algebraic equations of equilibrium are reduced to a series of linear equations which can be solved sequentially for the path derivatives. Each set of linear equations has the same basic non-singular matrix, making the procedure ideal for use with a digital computer.

A general theory is here presented, and this is subsequently illustrated firstly by an harmonic and secondly by a finite-element analysis of a simply-supported beam suffering large bending deflections under a lateral point load at its centre. An exact beam formulation is employed, and a *continuum* perturbation analysis is presented for comparison. The first seven path derivatives are evaluated and are observed to converge rapidly in each case to the continuum values which are then used to construct the load–deflection curve of the beam. The curve obtained depends on the independent variable employed in this final construction, and two curves are drawn using the load and its corresponding deflection respectively. Both curves are shown to be in good agreement with the known non-linear solution for central deflections up to about one-fifth of the length of the beam.

It is concluded that the perturbation analysis will be a useful tool in problems of moderate non-linearity.

2. GENERAL THEORY

Consider a discrete conservative structural system described by the well-behaved total potential energy function $V(Q_i, \Lambda)$, where Q_i represents a set of n generalized co-ordinates and Λ is a loading parameter [1].

The n equilibrium equations of the system are

$$V_i[Q_j, \Lambda] = 0 \tag{1}$$

where a subscript on V denotes partial differentiation with respect to the corresponding generalized coordinate. These equations define a series of equilibrium paths in the $(n+1)$-dimensional $(\Lambda - Q_i)$-space, and we shall here focus attention on the path emerging from the unloaded state $Q_i = \Lambda = 0$, writing this path in the parametric form

$$Q_i = Q_i(s), \qquad \Lambda = \Lambda(s). \tag{2}$$

Here s might represent any suitable parameter defining progress along the path, but we shall in fact assume that s is to be equated to one of the basic variables, Q_i or Λ, the formulation naturally taking a particularly simple and symmetric form if s is equated to the loading parameter Λ.

The equilibrium equations must be satisfied at every point on the equilibrium path, so we can substitute (2) into (1) to give the n equilibrium equations

$$V_i[Q_j(s), \Lambda(s)] = 0. \tag{E_0}$$

The left-hand side of each equation is now an implicit function of s, so we can differentiate each equation with respect to s as many times as we please. Differentiating once we have

$$V_{ij}\dot{Q}_j + V_i'\dot{\Lambda} = 0 \tag{E_1}$$

where a dot denotes differentiation with respect to s, a prime denotes differentiation with respect to Λ, and the dummy-suffix summation convention is employed with all summations ranging from one to n. Differentiating a second time we have

$$(V_{ijk}\dot{Q}_k + V_{ij}'\dot{\Lambda})\dot{Q}_j + V_{ij}\ddot{Q}_j + (V_{ij}'\dot{Q}_j + V_i''\dot{\Lambda})\dot{\Lambda} + V_i'\ddot{\Lambda} = 0, \tag{E_2}$$

etc., giving us the equations $E_1, E_2, E_3, E_4, \ldots$.

These equations can be evaluated at the unloaded state $Q_i = \Lambda = 0$, and we shall refer to the evaluated equation $E_m|_0$ as the mth-order equilibrium equation. In this equation all path derivatives are evaluated at the unloaded state.

We now allow one of the basic variables, Q_i or Λ, to be independent by equating it to s, and we correspondingly replace its first derivative with respect to s by unity, and its higher derivatives with respect to s by zero. The ordered equations $E_m|_0$ can then be solved sequentially as a system of linear equations.

Thus $E_1|_0$ represents a set of linear equations in the remaining first derivatives, and can readily be solved. These first derivatives can be substituted into the second-order equilibrium equation $E_2|_0$ which now represents a set of linear equations in the remaining second derivatives which are thus readily obtained. The known first and second derivatives can be substituted into the third-order equilibrium equation $E_3|_0$ which now represents a set of linear equations in the remaining third derivatives, which are likewise readily obtained. Clearly all the path derivatives can be obtained sequentially in this manner, and we see that each set of linear equations has the same basic matrix which will normally be non-singular.

As observed before, the equations take a particularly symmetric form if we equate s to the loading parameter Λ, and in this case we see that the determinant of each equation $E_m|_0$ is simply $|V_{ij}(0,0)|$. The quadratic coordinate form corresponding to $V_{ij}(0,0)$ is assumed to be positive definite, and it follows that there will be a unique solution for all the path derivatives $\dot{Q}_i|_0, \ddot{Q}_i|_0, \dddot{Q}_i|_0, \ldots$. There will then be a unique equilibrium path emerging from the unloaded state.

Having evaluated the path derivatives, we can finally construct the parametric equations of the equilibrium path in the series form

$$\left.\begin{aligned} Q_i(s) &= \dot{Q}_i(0)s+\tfrac{1}{2}\ddot{Q}_i(0)s^2+\tfrac{1}{6}\dddot{Q}_i(0)s^3+\ldots \\ \Lambda(s) &= \dot{\Lambda}(0)s+\tfrac{1}{2}\ddot{\Lambda}(0)s^2+\tfrac{1}{6}\dddot{\Lambda}(0)s^3+\ldots \end{aligned}\right\} \qquad (3)$$

A change in the independent variable is now readily effected, so the independent variable employed in the analysis itself has no lasting significance. The variable can be chosen for purely analytical convenience, and the final results can be readily re-cast in terms of any other suitable variable. The analysis itself has thus some desirable flexibility, but in the subsequent re-casting the choice of independent variable becomes important.

Consider for example the series expansion

$$y(x) = y_x(0)x+\tfrac{1}{2}y_{xx}(0)x^2+\tfrac{1}{6}y_{xxx}(0)x^3+\ldots$$

which is readily inverted to give

$$x(y) = x_y(0)y+\tfrac{1}{2}x_{yy}(0)y^2+\tfrac{1}{6}x_{yyy}(0)y^3+\ldots,$$

the first m derivatives of x with respect to y depending on the first m derivatives of y with respect to x (see for example [5], p. 11). If we wish to plot a graph of y against x the first m terms of the $y(x)$ expansion will naturally not yield the same curve as the first m terms of the $x(y)$ expansion. Both curves will normally differ from the true curve represented by $y(x)$ or $x(y)$ although they will both have correct values for the first m derivatives at the origin. The two power series may moreover have quite different ranges of convergence.

It follows that the choice of the independent variable acquires a real significance when an approximate solution is finally constructed from the path derivatives.

Special form

We shall now consider an important specialization of the preceding general theory which can be made when the original total potential energy function $V(Q_i, \Lambda)$ is linear in the loading parameter Λ.

When this linearity holds we shall, for purely semantic reasons, replace Λ by P and then write

$$V(Q_i, P) = U(Q_i)-P\mathscr{E}(Q_i). \qquad (4)$$

Here the function $U(Q_i)$ can be regarded as a *generalized strain energy*, while P can be regarded as the magnitude of a *generalized force* acting through the *generalized displacement* $\mathscr{E}(Q_i)$.

The n equilibrium equations are now

$$U_i(Q_j) = P\mathscr{E}_i(Q_j) \tag{5}$$

and proceeding as before the ordered equilibrium equations can be written in the evaluated form

$$U_{ij}\dot{Q}_j|_0 = \dot{P}\mathscr{E}_i|_0, \qquad (E_1|_0)$$

$$U_{ijk}\dot{Q}_k\dot{Q}_j + U_{ij}\ddot{Q}_j|_0 = \ddot{P}\mathscr{E}_i + 2\dot{P}\mathscr{E}_{ij}\dot{Q}_j|_0, \qquad (E_2|_0)$$

etc., where $E_1|_0$ is seen to be the well-known equation of small-deflection linear analysis.

These special equations can of course be obtained directly from the preceding general equations if we observe that

$$\left.\begin{aligned} V_{ij}(0,0) &\equiv U_{ij}(0) \\ V_i'(0,0) &\equiv -\mathscr{E}_i(0), \text{ etc.} \end{aligned}\right\} \tag{6}$$

Path derivatives of $\mathscr{E}$ can finally be obtained, if required, from the equations

$$\left.\begin{aligned} \dot{\mathscr{E}} &= \mathscr{E}_i\dot{Q}_i \\ \ddot{\mathscr{E}} &= \mathscr{E}_{ij}\dot{Q}_i\dot{Q}_j + \mathscr{E}_i\ddot{Q}_i, \text{ etc.} \end{aligned}\right\} \tag{7}$$

3. BEAM FORMULATION

The perturbation analysis will be illustrated by studies of a simply-supported beam suffering large bending deflections under a central point load, and we begin by writing down an exact large-deflection formulation for the beam.

Consider the beam of Fig. 1, of length L, simply-supported as shown, and loaded by the lateral force P which retains its magnitude and direction as the beam deflects. The

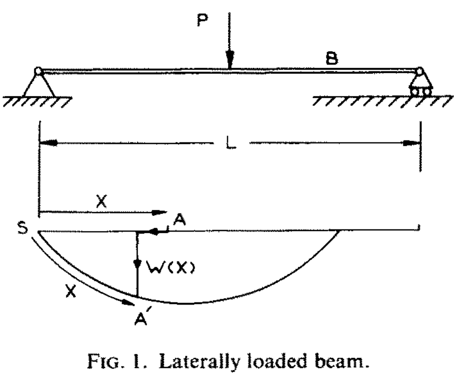

FIG. 1. Laterally loaded beam.

beam is assumed to be axially inextensional, and the relevant bending stiffness is denoted by B.

Point A of the beam originally distance X from the left-hand support is displaced to A', and this displacement is resolved into an unspecified horizontal component and a vertical component W as shown. The centre-line being inextensional, the arc-length SA' is equal to X, and the deflected form of the beam is totally specified by the function $W(X)$ where X ranges from 0 to L.

The curvature of an element is given by

$$\chi = \frac{\mathrm{d}}{\mathrm{d}X}\sin^{-1} W_X = W_{XX}(1-W_X^2)^{-\frac{1}{2}} \tag{8}$$

where subscripts on W denote differentiation with respect to X, so we have the strain energy functional

$$\left.\begin{aligned} U &= \tfrac{1}{2}B\int_0^L \chi^2\,\mathrm{d}X \\ &= \tfrac{1}{2}B\int_0^L W_{XX}^2(1-W_X^2)^{-1}\,\mathrm{d}X \end{aligned}\right\} \tag{9}$$

while the corresponding deflection of the load P is given by

$$\mathscr{E} = W(L/2). \tag{10}$$

The total potential energy of the system is thus given by

$$V = U - P\mathscr{E}. \tag{11}$$

Non-dimensionalizing now with respect to an unspecified length D, we write

$$\left.\begin{aligned} w &= W/D, \qquad \epsilon = \mathscr{E}/D, \qquad x = X/D, \\ v &= DV/B, \qquad u = DU/B, \qquad p = PD^2/B, \end{aligned}\right\} \tag{12}$$

to obtain the new energy expression

$$\left.\begin{aligned} v &= u - p\epsilon \\ &= \tfrac{1}{2}\int_0^{L/D} \ddot{w}^2(1-\dot{w}^2)^{-1}\,\mathrm{d}x - pw\left(\frac{L}{2D}\right), \end{aligned}\right\} \tag{13}$$

where a dot on w denotes differentiation with respect to x. It will be convenient to assign different values to D/L in the various subsequent analyses, and it is felt that this will give rise to no confusion, although naturally some care must be exercised in correlating the results of the various sections.

The EULER equation of the beam is given by

$$-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial F}{\partial \dot{w}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial F}{\partial \ddot{w}} = 0 \tag{14}$$

where $F(\ddot{w}, \dot{w})$ is the integrand of (13), and this can be written in the form

$$\begin{aligned} &\ddddot{w}[1+\dot{w}^2+\dot{w}^4+\ldots] \\ &\quad + 4\dddot{w}\ddot{w}\dot{w}[1+2\dot{w}^2+3\dot{w}^4+\ldots] \\ &\quad + \ddot{w}^3[1+6\dot{w}^2+15\dot{w}^4+\ldots] = 0. \end{aligned} \tag{15}$$

For this beam we now seek to relate the load parameter p to the deflection parameter ϵ by means of the power series

$$p(\epsilon) = c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + \ldots \tag{16}$$

$$\epsilon(p) = d_1 p + d_2 p^2 + d_3 p^3 + \ldots. \tag{17}$$

4. CONTINUUM ANALYSIS

Before presenting the Rayleigh–Ritz perturbation analyses, it seems appropriate to present here for comparison a continuum perturbation analysis of the beam.

In this non-linear continuum analysis it will be convenient to utilize the symmetry of the problem by considering only the left-hand side of the beam and setting $D = L/2$ so that x ranges from 0 to 1 over this half. We shall moreover replace the dead load P by the rigid imposition of a displacement $\mathscr{E}$, so that the continuum problem is specified by the EULER equation (15) with $0 < x < 1$ and the boundary conditions

$$w(0) = \ddot{w}(0) = \dot{w}(1) = 0, \qquad w(1) = \epsilon. \tag{18}$$

Choosing ϵ as our perturbation parameter, we now start the perturbation analysis by writing

$$\left.\begin{aligned} &w(x) = w_1(x)\epsilon + w_2(x)\epsilon^2 + w_3(x)\epsilon^3 + \ldots \\ &\text{so that} \\ &\dot{w}(x) = \dot{w}_1(x)\epsilon + \dot{w}_2(x)\epsilon^2 + \dot{w}_3(x)\epsilon^3 + \ldots \\ &\text{etc.} \end{aligned}\right\} \tag{19}$$

Substituting these series forms into the EULER equation (15) and writing this equation as a power series in ϵ, we see that we can equate the coefficients to zero to generate an ordered series of equilibrium equations, the boundary conditions of which can be written as

$$\begin{aligned} &w_i(0) = \ddot{w}_i(0) = \dot{w}_i(1) = 0 \quad \text{for all } i, \\ &w_1(1) = 1, \qquad w_s(1) = 0 \quad \text{for } s \neq 1. \end{aligned} \tag{20}$$

These equilibrium equations are sequentially linear and are readily solved for the $w_i(x)$, the first five of these functions being determined in the present work.

Thus equating the coefficient of ϵ to zero gives us the first-order equilibrium equation

$$\ddddot{w}_1 = 0 \tag{21}$$

which is the well-known small-deflection equation and yields with the boundary conditions (20) the solution

$$w_1(x) = -\tfrac{1}{2}x^3 + \tfrac{3}{2}x. \tag{22}$$

Equating the coefficient of ϵ^2 to zero gives us the second-order equilibrium equation $\ddddot{w}_2 = 0$ which with the boundary conditions (20) gives us the expected result $w_2(x) = 0$. The third-order equation is

$$\ddddot{w}_3 + 4\dddot{w}_1\ddot{w}_1\dot{w}_1 + \ddot{w}_1^3 = 0 \tag{23}$$

giving

$$w_3(x) = \tfrac{1}{280}\{27x^7 - 126x^5 + 171x^3 - 72x\}, \tag{24}$$

the fourth-order equation yields the expected result $w_4(x) = 0$ while the fifth-order problem yields

$$w_5(x) = \tfrac{9}{1120}\{-0{\cdot}573x^{11} + 8{\cdot}00x^9 - 26{\cdot}3x^7 + 20{\cdot}4x^5 + 8{\cdot}96x^3 - 10{\cdot}5x\}. \tag{25}$$

The force necessary to impose the central deflection is given from virtual work considerations by

$$p(\epsilon) = \frac{\partial}{\partial \epsilon}\int_0^1 \ddot{w}^2(1-\dot{w}^2)^{-1}\,\mathrm{d}x \tag{26}$$

and writing

$$p(\epsilon) = c_1\epsilon + c_2\epsilon^2 + c_3\epsilon^3 + \ldots \tag{27}$$

we can substitute for $w(x)$ and equate coefficients to obtain the c_i. These, like the $w_i(x)$, are zero for i even, and for i odd we have

$$\left.\begin{aligned} c_1 &= 2\int_0^1 (\ddot{w}_1)^2\,\mathrm{d}x = 6 \\ c_3 &= 4\int_0^1 [2\ddot{w}_3\ddot{w}_1 + (\ddot{w}_2)^2 + (\ddot{w}_1\dot{w}_1)^2]\,\mathrm{d}x = \tfrac{216}{35} \end{aligned}\right\} \tag{28}$$

etc., these coefficients being tabulated in Table 1 together with the readily derived inverse coefficients d_i.

TABLE 1. PERTURBATION COEFFICIENTS FROM THE CONTINUUM ANALYSIS ($D = L/2$)

c_1	c_3	c_5	c_7
6	6·17	7·18	8·59

d_1	d_3	d_5	d_7
1/6	$-4{\cdot}76\times10^{-3}$	$2{\cdot}54\times10^{-4}$	$-1{\cdot}66\times10^{-5}$

We see that the immediate expression for c_r will normally involve all w_i up to $i = r$, but since $\int_0^1 \ddot{w}_1\ddot{w}_s\,\mathrm{d}x$ vanishes identically for $s \neq 1$ by virtue of the boundary conditions (20), the dependency on w_r is illusory. In the present work we have thus evaluated the first seven c_i on the basis of the first five $w_i(x)$.

5. HARMONIC ANALYSIS

We shall now illustrate the general perturbation theory of Section 2 by an harmonic analysis of the beam, considering the whole beam and setting $D = L$ so that x ranges from 0 to 1.

Employing the symmetry of the beam we thus write

$$w = \sum_{i=1,2}^{i=n} Q_i \sin I\pi x \tag{29}$$

where $I \equiv 2i-1$, and substituting this form for $w(x)$ into the energy functional (13) gives us an algebraic energy function of the form

$$\left.\begin{aligned} v(Q_i, p) &= u(Q_i) - p\epsilon(Q_i) \\ &= \{\tfrac{1}{2}u_{ij}(0)Q_iQ_j + \tfrac{1}{24}u_{ijkl}(0)Q_iQ_jQ_kQ_l + \ldots\} \\ &\quad - p\epsilon_i(0)Q_i \end{aligned}\right\} \tag{30}$$

the coefficients being given by

$$\epsilon_i(0) = \sin\frac{I\pi}{2} = (-1)^{i+1} \tag{31}$$

$$\left.\begin{aligned} u_{ij}(0) &= \pi^4 I^2 J^2 \int_0^1 \sin I\pi x \sin J\pi x \,\mathrm{d}x \\ &= 0 \qquad \text{for } i \neq j \\ &= \tfrac{1}{2}\pi^4 I^4 \quad \text{for } i = j \end{aligned}\right\} \tag{32}$$

$$u_{ijkl}(0)Q_iQ_jQ_kQ_l = 12\pi^6 \sum\sum\sum\sum I^2J^2KLQ_iQ_jQ_kQ_l \int_0^1 \sin I\pi x \sin J\pi x \cos K\pi x \cos L\pi x \,\mathrm{d}x \tag{33}$$

etc.

We now have an energy function with the form of the special function (4) of the general theory, so the general analysis of Section 2 can be employed directly, and taking p as the independent variable we note that the diagonalization of $u_{ij}(0)$ allows the direct solution of each equilibrium equation $E_m|_0$. Thus $E_1|_0$ gives for all i,

$$\tfrac{1}{2}\pi^4 I^4 \dot{Q}_i = (-1)^{i+1} \tag{34}$$

so that

$$\dot{Q}_i = \frac{2}{\pi^4}\frac{(-1)^{i+1}}{I^4} \tag{35}$$

and

$$\left.\begin{aligned} \dot{\epsilon} = d_1 &= \frac{2}{\pi^4}\sum_{i=1}^{i=n} I^{-4} \\ &= \frac{2}{\pi^4}\left\{1 + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right\} \\ &= \frac{1}{48} \quad \text{(see [6], p. 64).} \end{aligned}\right\} \tag{36}$$

The second equation $E_2|_0$ gives $\ddot{Q}_i = \ddot{\epsilon} = 0$, and the third equation yields

$$\begin{aligned} \dddot{\epsilon} &= 6d_3 \\ &= -\frac{192}{\pi^{10}}\sum_1^n\sum_1^n\sum_1^n\sum_1^n \frac{(-1)^{i+j+k+l}}{I^2J^2K^3L^3}\int_0^1 \sin I\pi x \sin J\pi x \cos K\pi x \cos L\pi x \,\mathrm{d}x. \end{aligned} \tag{37}$$

Higher derivatives are readily written down, and the first seven have been evaluated on a computer, the corresponding series converging rapidly to the continuum solutions with increasing n.

6. FINITE ELEMENT ANALYSIS

As a second illustration of the general perturbation theory of Section 2 we shall now outline a finite-element analysis of the same beam using a kinematically-admissible displacement field. When used with such a field, the finite-element method can be viewed advantageously as a straight-forward application of the Rayleigh–Ritz energy procedure with localized rather than with overall Rayleigh functions. The analysis will thus follow closely the Rayleigh–Ritz analysis of Section 5 with localized functions replacing the overall harmonic functions.

Considering then the displacement function $w(x)$ for the whole beam, we introduce stations along the x-axis as shown in Fig. 2, one station being at the loaded mid-point.

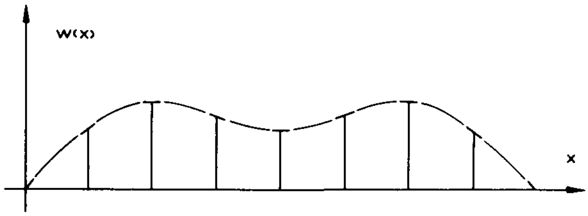

FIG. 2. Finite-element representation of $w(x)$.

Each internal station is given an arbitrary displacement w and an arbitrary first derivative $\dot{w}$, while corresponding to the boundary conditions of the problem the terminal stations are given an arbitrary first derivative but no displacement. The totality of the arbitrary displacements and derivatives represent the generalized coordinates of the beam and are denoted by Q_i. A third-order polynomial is now fitted in each region, giving an overall deflected form which is continuous in displacement and first derivative, but which admits discontinuities in its second derivative at the stations.

The Rayleigh–Ritz aspect of this fitting procedure is revealed if we now observe that the overall displacement function has the form

$$w(x) = \sum Q_i f_i(x)$$

where the $f_i(x)$ are localized functions associated with the stations [7, 8].

Substituting this expression for $w(x)$ into the energy functional (13) gives us an algebraic energy function with the form of that of the harmonic analysis (30), so the special form of the general theory can again be directly employed. Now, however, the quadratic form associated with $u_{ij}(0)$ is not diagonal, so the series of linear problems must be solved numerically on a computer. The first seven have been solved in the present study, the displacement of the central station being used as the independent variable. The first two path derivatives were found to agree with the continuum values for a single internal station, while the higher path derivatives were observed to converge rapidly to the continuum values with increasing number of stations.

More details of this finite-element study are to be presented in a further paper [9].

7. PERTURBATION SOLUTIONS

The two Rayleigh–Ritz analyses, namely the harmonic analysis and the finite-element analysis, yield perturbation coefficients c_i and d_i which converge rapidly to those of the continuum analysis, so we shall here focus attention on the latter which are recorded in Table 1. These coefficients correspond to the expansions of $p(\epsilon)$ and $\epsilon(p)$ in equations (16) and (17) and the numerical values of the table are associated with a choice of $D = L/2$.

Taking firstly ϵ as our independent variable and truncating the $(p)\epsilon$ expansion after one, three, five and seven terms yields the curves I, III, V and VII of Fig. 3 in which B represents the known non-linear beam solution [10]. The perturbation solutions are seen to approach

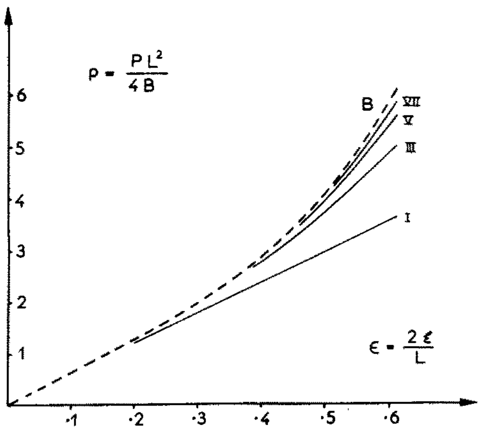

FIG. 3. Perturbation solutions with ϵ as the independent variable.

this exact solution monotonically, the seventh-order solution yielding close agreement with B for central deflections up to one quarter of the length of the beam.

Taking secondly p as our independent variable and truncating the $\epsilon(p)$ expansion after one, three, five and seven terms yields the curves I, III, V and VII of Fig. 4, in which B

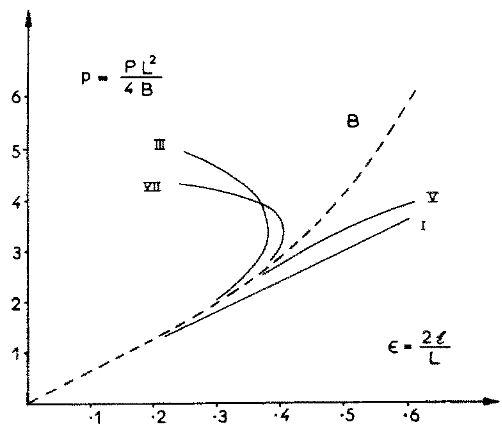

FIG. 4. Perturbation solutions with p as the independent variable.

again represents the known non-linear solution. The perturbation solutions are now seen to band this exact solution in an oscillatory fashion with an apparent divergence at large deflections.

Remembering that these two figures represent the results of the same perturbation analysis, the contrast is suprising indeed, and emphasizes the key role of the final expansion.

8. CONCLUSIONS

The perturbation analysis of the equilibrium path of a discrete non-linear structural system generates a system of linear equations which can be solved sequentially for the path derivatives in the unloaded state. Each set of linear equations has the same basic non-singular matrix, and the method is thus ideally suited for use with a digital computer.

It is felt that the method will be particularly useful for supplying a *first estimate* of the degree of nonlinearity of a problem, and that it will supply a satisfactory *solution* to problems exhibiting a moderate degree of nonlinearity. In highly non-linear problems convergence difficulties may arise and a direct solution of the non-linear algebraic equations (using for example the Newton–Raphson procedure) seems more appropriate.

In the perturbation analysis itself the independent variable should be chosen for purely analytical convenience, the results of the analysis being readily re-cast in terms of any other suitable variable. The final choice of independent variable is seen to be important, but it seems unlikely that any general rules can be devised to guide this choice.

An extension of the perturbation approach to the estimation of elastic critical loads has recently been outlined [11] following an idea of Masur and Schreyer [12], while the use of a perturbation analysis in problems of initial post-buckling is an obvious extension of the work of Sewell and Thompson.

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(*Received* 20 *November* 1967; *revised* 9 *February* 1968)

Абстракт—Представляется общая форма анализа возмущений для дискретных нелинейных систем. Это обобщает систему линейных уравнений, которые можно решать последовательно для траекторий производных в ненагруженном состоянии. Каждая система линейных уравнений имеет основную, несингулярную метрицу. Метод расчета очень пригоден для использования на вычислительных машинах. Общая теория иллюстрируется, во первых, гармоническим анализом и, во вторых, анализом конечного элемента балки испытываемой большим изгибам.

Для сравнения представляется точное решение балкм и метод непреривных возмущений. Определяются иервые семь траекторий. Наблюдается быструю сходимость в каждом случае, по сравнению со значениями из непрерывного анализа, которые используются для постоения характеристик нагрузка-прогиб в балке: оказывается, что выбор независимой переменной в этом остаточном процессе очена важен. Получается хорошее согласие с известными нелинейными решениями. В конце концов оказывается, что анализ возмущений будет пригоден к задачам, обладающим средней нелинейностью.